

A New Look at the Arcsine Law and “Quantum-Classical Correspondence”

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February 8, 2012

Abstract

We prove that the Arcsine law as the “time-averaged distribution” for classical harmonic oscillator emerges from the distributions for quantum harmonic oscillators in terms of noncommutative algebraic probability. This is nothing but a simple and rigorous realization of “Quantum-Classical Correspondence” for harmonic oscillators.

1 Introduction

The normalized Arcsine law μ_{As} is the probability distribution on \mathbb{R} with support $[-\sqrt{2}, \sqrt{2}]$ defined as

$$\mu_{As}(dx) = \frac{1}{\pi} \frac{dx}{\sqrt{2-x^2}},$$

whose n -th moment $M_n := \int_{\mathbb{R}} x^n \mu_{As}(dx)$ is given by

$$M_{2m+1} = 0, \quad M_{2m} = \frac{1}{2^m} \binom{2m}{m}.$$

In this case, the moment sequence $\{M_n\}$ characterizes μ_{As} (“the moment problem is deterministic”).

The distribution μ_{As} often appears in classical probability theory. In the noncommutative context, it is also known as the limit distribution for

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“monotone central limit theorem” ([3], a simple proof is found in [4]). Here we discuss another aspect of this distribution: the relationship with the notion of “Classical harmonic oscillator”.

Let $x(t) = A \sin t$ be a harmonic oscillator with amplitude A . Then it is easy to see that “the time-averaged distribution μ of position x ” has the form

$$\mu(dx) = C \frac{dx}{\sqrt{A^2 - x^2}}$$

where C denotes the normalizing constant. In $A = \sqrt{2}$ case, $\mu = \mu_{As}$.

Then a question arises: Is it possible to see whether and in what meaning the “Quantum-Classical Correspondence” holds for harmonic oscillators? This question, which is related to fundamental problems in Quantum theory and asymptotic analysis [2], is analyzed and generalized from the viewpoint of noncommutative algebraic probability with quite a simple combinatorial argument.

2 Basic Notions

Let \mathcal{A} be a $*$ -algebra. We call a linear map $\varphi : \mathcal{A} \rightarrow \mathbb{C}$ a state on \mathcal{A} if it satisfies

$$\varphi(1) = 1, \quad \varphi(a^*a) \geq 0.$$

A pair (\mathcal{A}, φ) of a $*$ -algebra and a state on it is called an algebraic probability space. Here we adopt a notation for a state $\varphi : \mathcal{A} \rightarrow \mathbb{C}$, an element $X \in \mathcal{A}$ and a probability distribution μ on \mathbb{R} .

Notation 2.1. $X \sim_{\varphi} \mu \iff \varphi(X^m) = \int_{\mathbb{R}} x^m \mu(dx)$ for all $m \in \mathbb{N}$.

Remark 2.2. Existence of μ for X which satisfies $X \sim_{\varphi} \mu$ always holds. The uniqueness of such μ holds if the moment problem is deterministic.

Definition 2.3 (Quantum harmonic oscillator). A quantum harmonic oscillator is a triple $(\Gamma(\mathbb{C}), a, a^*)$ where $\Gamma(\mathbb{C})$ is a Hilbert space $\Gamma(\mathbb{C}) := \bigoplus_{n=0}^{\infty} \mathbb{C}\Phi_n$ with inner product given by $\langle \Phi_n, \Phi_m \rangle = \delta_{n,m}$, and a, a^* are operators defined as follows:

$$a\Phi_0 = 0, \quad a\Phi_n = \sqrt{n}\Phi_{n-1} (n \geq 1)$$

$$a^*\Phi_n = \sqrt{n+1}\Phi_{n+1}$$

Let \mathcal{A} be the $*$ -algebra generated by a , and $\varphi_n(\cdot)$ be the state defined as $\varphi(\cdot) := \langle \Phi_n, (\cdot) \Phi_n \rangle$. Then $(\mathcal{A}, \varphi_n(\cdot))$ is an algebraic probability space. It is well known that

$$X := \frac{1}{\sqrt{2}}(a + a^*)$$

represents the “position” observable and that

$$X \sim_{\varphi_0} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx.$$

That is, in $n = 0$ case, the distribution of position operator is Gaussian.

On the other hand, the asymptotic behavior of the distribution of position operator as n tends to infinity is quite nontrivial.

3 Emergence of Arcsin law

Theorem 3.1. *Let μ_N be a probability distribution on \mathbb{R} such that*

$$\frac{X}{\sqrt{N+1}} \sim_{\varphi_N} \mu_N.$$

Then μ_N weakly converges to μ_{As} .

Proof. We only have to prove moment convergence.

First we can easily prove that

$$\varphi_N\left(\left(\frac{X}{\sqrt{N+1}}\right)^{2m+1}\right) = \langle \Phi_N, \left(\frac{a+a^*}{\sqrt{2(N+1)}}\right)^{2m+1} \Phi_N \rangle = 0$$

since $\langle \Phi_N, \Phi_M \rangle = 0$ when $N \neq M$. To consider the moments of even degrees, we introduce the following notations:

- $\Lambda^{2m} := \{\text{maps from } \{1, 2, \dots, 2m\} \text{ to } \{1, *\}\},$
- $\Lambda_m^{2m} := \{\lambda \in \Lambda^{2m}; |\lambda^{-1}(1)| = |\lambda^{-1}(*)| = m\}.$

Then

$$\begin{aligned} \varphi_N\left(\left(\frac{X}{\sqrt{N+1}}\right)^{2m}\right) &= \langle \Phi_N, \left(\frac{a+a^*}{\sqrt{2(N+1)}}\right)^{2m} \Phi_N \rangle \\ &= \frac{1}{2^m} \sum_{\lambda \in \Lambda^{2m}} \frac{1}{(N+1)^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \dots a^{\lambda_{2m}} \Phi_N \rangle \\ &= \frac{1}{2^m} \sum_{\lambda \in \Lambda_m^{2m}} \frac{1}{(N+1)^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \dots a^{\lambda_{2m}} \Phi_N \rangle \\ &\rightarrow \frac{1}{2^m} |\Lambda_m^{2m}| = \frac{1}{2^m} \binom{2m}{m} \quad (N \rightarrow \infty) \end{aligned}$$

because

$$N \cdots (N - m + 1) \leq \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \leq (N + 1) \cdots (N + m)$$

for sufficiently large N and then

$$\frac{1}{(N + 1)^m} \langle \Phi_N, a^{\lambda_1} a^{\lambda_2} \cdots a^{\lambda_{2m}} \Phi_N \rangle \rightarrow 1 \quad (N \rightarrow \infty).$$

This completes the proof. □

Remark 3.2. The theorem above can be extended to the cases for q -Fock spaces ($-1 < q \leq 1$), which are typical example of “interacting Fock spaces [1]”, if we just replace integer $N + 1$ by

$$(N + 1)_q := 1 + q + q^2 + \cdots + q^N.$$

The proof is quite similar and we omit it here.

4 Summary and prospects

As we have stated, the Arcsine law as the “time-averaged distribution” for classical harmonic oscillator emerges from the distributions for quantum harmonic oscillators. This is nothing but a noncommutative algebraic realization of “Quantum-Classical Correspondence” for harmonic oscillators. The “time averaged” nature is deeply related to the notion of Bohr’s “complementarity” for energy and time. Starting from energy eigenstates, one cannot obtain the classical harmonic oscillator itself but “time averaged distribution” of it.

Mathmatically, the result above also shows an important aspect of the Arcsine law as the universal distribution for many kinds of “interacting Fock spaces” [1] which is deeply connected to the theory of orthogonal polynomials (This point of view is due to Professor Bożejko). The condition for interacting Fock spaces (orthogonal polynomials) from which the Arcsine law emerges as the “high-energy limit distribution” should be discovered.

Acknowledgments

The author would like to thank Prof. Marek Bożejko for interests and comments. He is greatly indebted to Prof. Izumi Ojima and Mr. Kazuya Okamura for discussions on “Quantum-Classical Correspondence”.

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